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Lesson 13: Integration (Part 1)

Antiderivatives

- This lesson is an introduction to integral calculus. As mentioned earlier, the primary goal of integral calculus is to find the function, given its derivative.
  - A function F(x) is called an **antiderivative** of function f(x) on an interval I, if F'(x) = f(x) for all x in I.
  - So given f(x), the goal in integral calculus is to find the antiderivative F(x). However, finding the antiderivative is often a more difficult problem than finding the derivative; and for many functions f(x), we cannot express F(x) using a simple formula.
  - As we will soon see, antiderivatives allow us to evaluate definite integrals, which are essentially sums with an infinite number of terms, where each term approaches zero. We will use definite integrals to compute the area between curves, but the concept is widely used in various problems of science and engineering.
- From Corollary 2 of the Mean Value theorem, we know that all antiderivatives of f(x) differ by a constant.
  - Hence if F(x) is an antiderivative of f(x), then F(x) + C, where C is a constant represents the set of all antiderivatives. It represents a family of functions whose graphs are vertically shifted with respect to each other.
  - The set of all antiderivatives of f(x) is also called the **indefinite integral** of f with respect to x and is denoted by $\int f(x)dx$. The symbol $\int$ is called the **integral sign** and f(x) is called the **integrand** and x is the **variable of integration**.

  Therefore $\int f(x)dx = F(x) + C$

Antiderivatives (continued)

Finding the indefinite integral of f(x) is equivalent to solving the below equation for the unknown variable (function) y.

$$\frac{dy}{dx} = f(x)$$

The equation is a simple **differential equation**. If y_0 is given for a specific x_0, then it is called an **initial condition**; and it can be used to determine the constant in the indefinite integral (see below example).

A few simple (and obvious) rules for evaluating indefinite integrals are given below (in each case, you can differentiate the indefinite integral on the right to obtain the integrand).

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

If $F'(x) = f(x)$ and $G'(x) = g(x)$, then

$$\int \left[f(x) \pm g(x) \right] dx = F(x) \pm G(x) + C$$

$$\int kf(x)dx = kF(x) + C \quad (k \text{ is a constant})$$

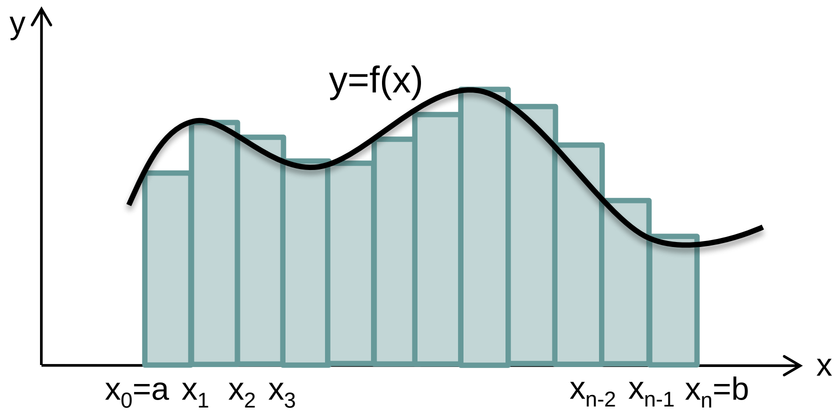
Example: Solve $\frac{dy}{dx} = 3x^2 + 4x + 1$ where y(0) = 1

$$y = \int (3x^2 + 4x + 1)dx = 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + x + C = x^3 + 2x^2 + x + C \quad \text{(using the rules given above)}$$

Substituting y(0) = 1 in the above expression gives 1 = 0 + C

$$\therefore y = x^3 + 2x^2 + x + 1$$

Finding areas



- We will now introduce the concept of definite integral using the standard area problem. Consider a non-negative function y = f(x). What is the area A between the graph of y = f(x), the x axis, the lines x = a and x = b?
- The method to find this is:
  - Divide the interval between $x_0=a$ and $x_n=b$ into n parts by choosing points $x_1, x_2 \ldots x_{n-1}$ between them such that $x_0 < x_1 < x_2 \ldots x_{n-1} < x_n$. The set of points $\{x_0, x_1, x_2, \ldots x_{n-1}, x_n\}$ is called a **partition** of the interval [a, b].
  - Because of the partition, the interval [a, b] is divided into n sub-intervals $[x_0, x_1]$, $[x_1, x_2]$ … $[x_{n-1}, x_n]$. The width of the ith sub-interval is $\Delta x_i = x_i - x_{i-1}$. The largest sub-interval width is called the **norm** of the partition.

Finding areas (continued)

- Choose a point in each sub-interval, e.g. c_i is the point chosen in the ith sub-interval $[x_{i-1}, x_i]$. So $x_{i-1} \le c_i \le x_i$.
- On each sub-interval, draw a rectangle whose width is the subinterval width and height is the value of the function f at the chosen point. e.g. the rectangle in the ith sub-interval has width Δx_i and height $f(c_i)$.
- The area in question (A) is approximated by the sum of the areas of the rectangles drawn, i.e. $A \cong f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \ldots + f(c_{n-1})\Delta x_{n-1} + f(c_n)\Delta x_n$
- This sum called the **Riemann sum of f on the interval [a, b]** can approach a limiting value as the norm approaches 0 (implies the number of sub-intervals approaches infinity). The limiting value is called the **definite integral of f over [a, b]**.

$$\text{The definite integral is written as } \int_a^b f(x)dx = \lim_{norm \to 0}\left(\sum_{i=1}^{n} f(c_i)\Delta x_i\right)$$

a is called the **lower limit** of integration and b is the **upper limit**. f(x) is the integrand and x is the variable of integration. The area A is <u>exactly</u> represented by the definite integral.

Note: The symbol ∑ called sigma is used to represent sums in a compact form. Following the sigma, the ith term in the sum is specified. "i" is called the index of summation and it specifies the term from where the summation starts (specified below the sigma, and is 1 above) and the term where the summation ends (specified above the sigma, and is n above).

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \underbrace{f(c_1)\Delta x_1}_{i=1} + \underbrace{f(c_2)\Delta x_2}_{i=2} + \ldots + \underbrace{f(c_{n-1})\Delta x_{n-1}}_{i=n-1} + \underbrace{f(c_n)\Delta x_n}_{i=n}$$

Finding areas (continued)

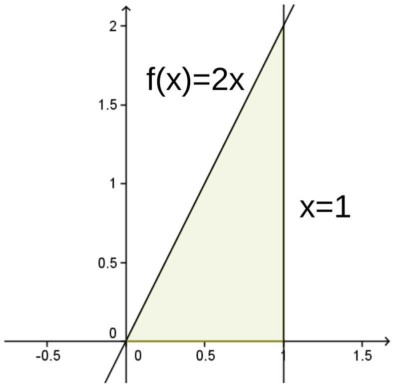
A function f(x) that is continuous on [a, b] has a definite integral over [a, b] and we say that the function is **integrable** over [a, b].

The function can have negative values as well. On intervals where f is negative, the sum represented by the definite integral gives the negative of the area.

If we divide $[a,\ b]$ into n equal sub-intervals, and we pick c as the left endpoint in each sub-interval,

then the summation $\sum_{i=1}^{n} f(c_i)\Delta x_i$ becomes $\dfrac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$

Example: Consider $f(x)=2x$ and calculate the area A between this function, the x-axis and the vertical lines $x=0$ and $x=1$.



From geometry, we know the area A= $\dfrac{1}{2}\times 1\times 2=1$

From the definite integral viewpoint, when we divide [0, 1] into n equal sub-intervals, the partition is $\{0,\ \dfrac{1}{n},\ \dfrac{2}{n},\ldots\dfrac{n-1}{n},\ 1\}$. We take the left endpoint of each sub-interval to calculate the area A.

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)=\frac{1}{n}\left(0+2.\frac{1}{n}+2.\frac{2}{n}+\ldots 2.\frac{n-1}{n}\right)=\frac{2}{n^2}\left(1+2+\ldots+n-1\right)=\frac{2}{n^2}\,\frac{(n-1)n}{2}=\frac{n-1}{n}$$

$$A=\int_0^1 2x\,dx=\lim_{n\to\infty}\frac{n-1}{n}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1 \text{ (agrees with the value from geometry, as it should).}$$

Some properties of definite integrals

$$\int_b^a f(x)dx = -\int_a^b f(x)dx \qquad \text{[Property 1]}$$

Using equal sub-intervals and left endpoint in the Riemann sum, we have

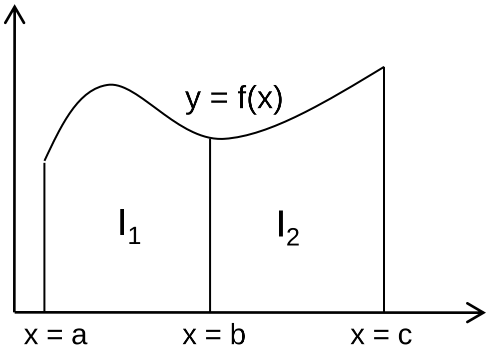
$$\int_b^a f(x)dx = \lim_{n\to\infty} \frac{a-b}{n} \sum_{i=0}^{n-1} f(x_i)$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=0}^{n-1} f(x_i)$$

Hence they are negative of each other (basically Δx in the sum changes sign).

If $a = b$, then the definite integral is 0.

$$\underbrace{\int_a^c f(x)dx}_{I} = \underbrace{\int_a^b f(x)dx}_{I_1} + \underbrace{\int_b^c f(x)dx}_{I_2} \qquad \text{[Property 2]}$$



Holds for any values of a, b and c (no particular ordering, e.g. a < b < c is implied here). The interval from a to b, and that from b to c can overlap. If it does, the contribution in the overlapping region from integrals I_1 and I_2 cancel as per Property 1. What remains is the sum over the a to c interval.

For a < b < c and f(x) non-negative, the interpretation is that the area given by integral I is the sum of the areas I_1 and I_2.

Properties of definite integrals (continued)

If M is the absolute maximum value of $f(x)$ in $[a,\ b]$ and m is the absolute minimum value of f, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a) \qquad \text{[Property 3]}$$

Proof: For any partition of $[a,\ b]$ and using the notations introduced earlier for Riemann sum

$$m(b-a) = m\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} m\Delta x_i \le \sum_{i=1}^{n} f(c_i)\Delta x_i \le \sum_{i=1}^{n} M\,\Delta x_i = M(b-a)$$

Therefore in the limit as the norm $\to 0$, the stated inequality follows.

Mean Value Theorem for definite integrals [Property 4]: If $f(x)$ is continuous in $[a,\ b]$, then for some point c in $[a,\ b]$, $\quad f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$

Proof: From the previous theorem, we have: $m \le \underbrace{\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx}_{I} \le M$

Note m and M are the minimum and maximum values respectively of f(x) in [a, b], so if $f(x_1) = m$ and $f(x_2) = M$, then x_1 and x_2 are in [a, b].

Since f(x) is continuous in [a, b], it is also continuous in the interval between x_1 and x_2, and from the Intermediate Value theorem, we know that f(x) takes on every value between m and M in [a, b].

Since integral I is between m and M, it should be equal to f(c) for some c in [a, b].

Properties of definite integrals (continued)

Also note $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = \dfrac{1}{a-b}\displaystyle\int_b^a f(x)dx$ [by Property 1]

$\therefore$ we can write the mean value theorem as $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx = f(c)$ for some c between a and b

(removing the restriction that $a < b$).

Average of function f over $[a,\ b]$ is defined as $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$

Rationale: Let $[a,\ b]$ be partitioned into n equal sub-intervals and c_i be any point in the ith subinterval.

Then average $\cong \dfrac{f(c_1)+f(c_2)+\ldots+f(c_{n-1})+f(c_n)}{n} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} f(c_i)$

But $\Delta x = \dfrac{b-a}{n}$. So average $= \dfrac{1}{b-a}\displaystyle\sum_{i=1}^{n} f(c_i)\Delta x$

When $\Delta x \to 0$ $(n \to \infty)$, and average becomes $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$

Fundamental theorem of calculus

As the name implies, the Fundamental theorem of calculus, is perhaps the most fundamental result in calculus; and it relates the definite integral of f(x) over [a, b] to the antiderivative of f. It consists of two parts, that are stated and proved now.

Thereom, Part 1: If $f(x)$ is continuous in $[a,\ b]$ and $F(x) = \int_a^x f(t)dt$, then $F(x)$ is differentiable on $(a,\ b)$ with $F'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$.

Proof: Let us start with the difference quotient of $F(x)$

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right) = \frac{1}{h}\int_x^{x+h} f(t)dt \quad \text{[By property 2 of definite integrals]}$$

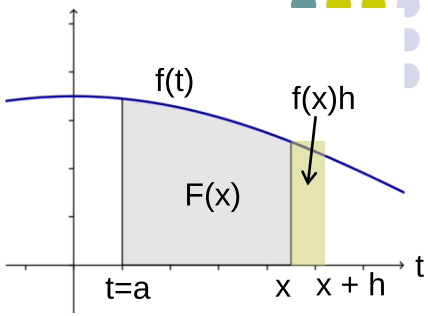
$\frac{1}{h}\int_x^{x+h} f(t)dt = f(c)$ for some c in the interval between x and $x+h$ [Mean Value Theorem for integrals]

Note h can be positive or negative.

$$\therefore F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(c) = f(x) \text{ since } f \text{ is continuous.}$$

At the endpoint x = a, the right hand derivative of F(x) is f(a), and at the endpoint x = b, the left hand derivative of F(x) is f(b). Because F(x) is differentiable, it is also continuous in [a, b].

Fundamental Theorem of Calculus (continued)



Part 1 of the theorem should be intuitive, from the diagram shown. F(x) is the area under f(t) from "a" to x, while F(x + h) is the area from "a" to x + h (h > 0). When h is small, F(x + h) – F(x) ≅ f(x)h (the area of the small rectangle). So we expect F'(x) = (F(x + h) – F(x)) / h = f(x).

Thereom Part 2: If $f(x)$ is continuous in $[a,\ b]$ and $F(x)$ is any antiderivative of f on $[a,\ b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof: $G(x) = \int_a^x f(t)dt$ is an antiderivative of $f(x)$ from Part 1 of this theorem.

Since all antiderivatives of f differ by a constant (from Corollary 2 of Mean Value Theorem), the most general antiderivative $F(x) = G(x) + C$

$$\therefore F(b) - F(a) = \left(G(b) + C\right) - \left(G(a) + C\right) = G(b) - G(a)$$

$$= \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt = \int_a^b f(x)dx$$

(Changing the variable of integration from t to x, doesn't change the definite integral).

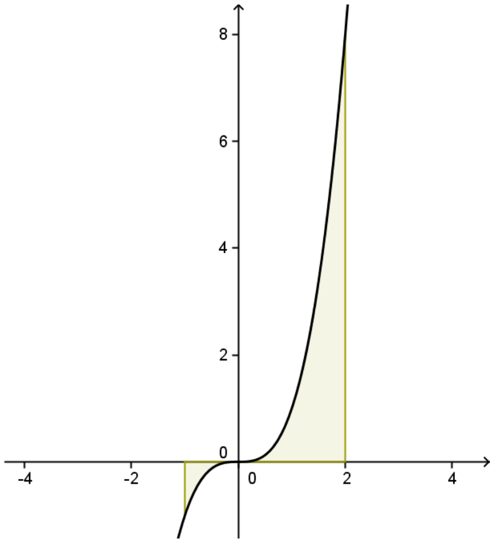
We usually write $\int_a^b f(x)dx = F(x)\Big|_a^b$ or $\left[F(x)\right]_a^b = F(b) - F(a)$

Example and substitution rule

Example: Find the area between the graph $f(x) = x^3$, the x-axis and the lines $x = -1$ and $x = 2$.

In $[-1,\ 0)$, $f(x) < 0$, hence the definite integral is the negative of the area.

$$\therefore \text{Area } A = -\int_{-1}^{0} x^3 dx + \int_{0}^{2} x^3 dx = -\left.\frac{x^4}{4}\right|_{-1}^{0} + \left.\frac{x^4}{4}\right|_{0}^{2} = -\frac{0-1}{4} + \frac{16-0}{4} = \frac{17}{4}$$



The **substitution rule** is a useful tool for evaluating indefinite integrals. It states that if u = g(x) is a differentiable function whose range is an interval I, and f(x) is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

So to evaluate ∫f(g(x))g'(x)dx, we do the following:

- Substitute u = g(x) and du = g'(x)dx in the integral to obtain ∫f(u)du.
- Integrate with respect to u
- Replace u by g(x) in the result.

Proof: If $F(x)$ is the antiderivative of $f(x)$, then by the chain rule

$$\frac{dF(g(x))}{dx} = F'(g(x))g'(x) = f(g(x))g'(x) \rightarrow \int f(g(x))g'(x)dx = F(g(x)) + C \quad (1)$$

Also $\int f(u)du = F(u) + C = F(g(x)) + C$ (as in 1) above).

Substitution rule (continued)

We introduced the concept of differential in an earlier lesson.

If ∫f(x)dx = F(x) + C, then the differential d(F(x)) = f(x)dx. So we can write ∫d(F(x)) = F(x) + C.

When u = g(x), its differential du = g'(x)dx. Substituting in ∫f(g(x))g'(x)dx, we get ∫f(u)du. As per the substitution rule, both integrals are equal. So we can interpret the whole expression under the integral sign, as well as dx and du, as differentials.

For a definite integral, the substitution rule is written as follows:

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Proof: If $F(x)$ is the antiderivative of $f(x)$, then

$$\int_a^b f(g(x))g'(x)dx = F(g(x))\Big|_a^b = F(g(b)) - F(g(a))$$

$$\int_{g(a)}^{g(b)} f(u)du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a)) \quad \text{(the two sides are equal).}$$

To evaluate a definite integral using the substitution rule, we proceed as we do for indefinite integrals. Additionally, we substitute for the upper and lower limits; when x = a, u = g(x) is g(a) and when x = b, u = g(b).

Examples

Evaluate $\int_0^1 x^3(1+x^4)^4\,dx$

Substitute $u=1+x^4$. Then $du=4x^3dx$. When $x=0,\ u=1$; and when $x=1,\ u=2$.

$$\therefore \int_0^1 x^3(1+x^4)^4\,dx = \frac{1}{4}\int_1^2 u^4du = \frac{1}{20}u^5\Big|_1^2 = \frac{31}{20}$$

Alternative method: Don't change the limits of integration, use substitution to only find the antiderivative.

So in $\int x^3(1+x^4)^4\,dx$, substitute $u=1+x^4$. Then $du=4x^3dx$, and we have

$$\int x^3(1+x^4)^4\,dx = \frac{1}{4}\int u^4du = \frac{u^5}{20}+C = \frac{(1+x^4)^5}{20}+C$$

$$\therefore \int_0^1 x^3(1+x^4)^4\,dx = \frac{(1+x^4)^5}{20}\Bigg|_0^1 = \frac{32-1}{20} = \frac{31}{20}$$

Differential equation with separable variables

There are many types of differential equations; here we consider the solution of a first order differential equation with separable variables. First order implies that we have only dy/dx in the equation, and no higher order derivatives like d^2y/dx^2.

By definition, a separable first order equation can be written as

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \text{(1). We can write this as } g(y)\frac{dy}{dx} = f(x), \text{ and to solve, we integrate both sides w.r.t. } x.$$

$$\int g(y)\frac{dy}{dx}dx = \int f(x)dx \rightarrow \int g(y)dy = \int f(x)dx$$

$$\rightarrow G(y) = F(x) + C \qquad \text{(2), where } G'(y) = g(y) \text{ and } F'(x) = f(x).$$

If $y = y_0$ for $x = x_0$ (initial condition), then $C = G(y_0) - F(x_0)$, and we can write 2) as

$$G(y) - G(y_0) = F(x) - F(x_0) \rightarrow \int_{y_0}^{y} g(y')dy' = \int_{x_0}^{x} f(x')dx' \qquad (3)$$

Note by definition, differential dy = (dy/dx)dx. Also by differentiating 2) implicitly w.r.t. x, we get 1). Therefore, we can solve 1), by treating dy/dx as a ratio, and separating the expressions for x and y.

Also form 3) incorporates the initial condition into it (we used x' and y' to distinguish the integration variable from the upper limit x and y respectively).

Example

Example: Solve the equation, $\frac{dy}{dx} = \frac{1}{2y(1+x^2)}$ with the initial condition $y(0) = 2.$

Solution: Separating the variables, we have

$$\int 2y\,dy = \int \frac{dx}{1+x^2} \to y^2 = \tan^{-1} x + C.$$

The initial condition gives $2^2 = 0 + C \to y^2 = \tan^{-1} x + 4.$

We have used the standard formula for the derivative of $\tan^{-1}x$ (given in an earlier lesson).

Also $m = \tan^{-1}0$ implies $\tan m = 0$ or $m = 0$.