# Lesson 11: Derivatives (Part 1)

# Introduction

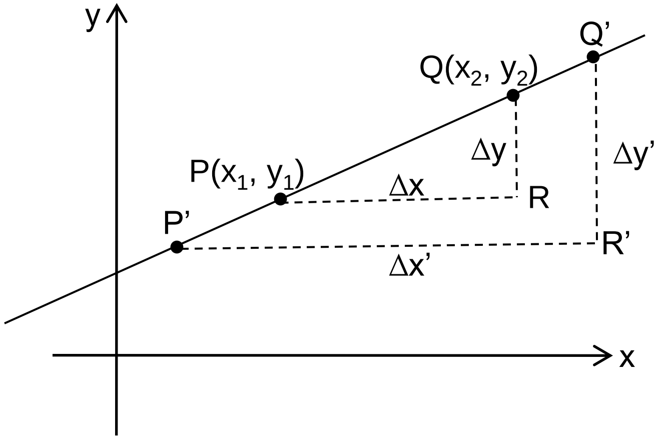
- We are now ready to look at differential calculus. There are essentially two parts to calculus: differential calculus and integral calculus.
  - **Differential calculus** is concerned with finding the derivative of functions (we will soon see what a derivative is).
  - **Integral calculus** is concerned with finding the function, given its derivative. As we will see later, this is necessary to evaluate definite integrals, which are essentially sums with an infinite number of terms, where each term approaches zero. Such sums arise in science and engineering, and you will often see them in your study of physics.
- To introduce the concept of derivative, let us go back to the concept of slope of a line, and tangent to a curve.

We know that the **slope** m of a line passing through points $(x_1, y_1)$ and $(x_2, y_2)$ is defined as

$m = \Delta y/\Delta x = (y_2 - y_1) / (x_2 - x_1) = \Delta y'/\Delta x'$ (see figure).

Slope is independent of points chosen, since triangles PQR and P'Q'R' are similar.

$\Delta y$ (read as "delta y") refers to the change in y as we go from point P to point Q, and $\Delta x$ refers to the <u>corresponding</u> change in x.

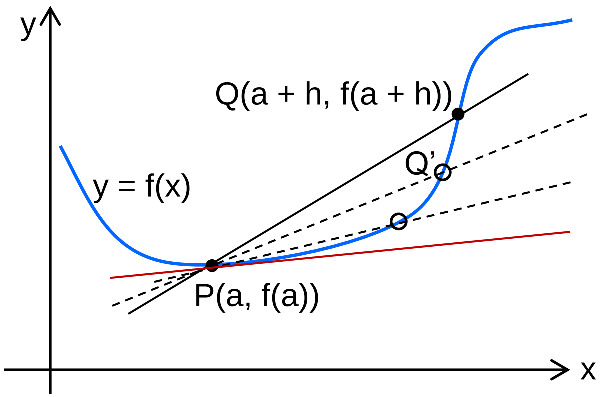


# Slope of a curve, and derivative

For the generic curve $y = f(x)$, slope depends on the points P and Q chosen (on the curve) to measure it. However it is useful to define slope at a point P(a, f(a)) as follows:

Draw a line through P and an adjacent point Q(a + h, f(a + h)); this line is called a **secant**.

$$\text{Slope of secant PQ} = \frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h}$$




**Slope of the curve** at point P is defined as the limit of the secant slope, when h approaches 0 (that is Q approaches P).

$$\text{Slope of curve} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

The limit is called the **derivative** of f(x) at x = a, denoted by f'(a). Note the limit is the usual two sided limit, hence h can be positive or negative.

The line through P(a, f(a)) with slope equal to f'(a) is called a **tangent** to the curve at P. The secant PQ becomes the tangent at P, when Q moves closer and closer to P (along the curve), so as to merge into P.

$$\frac{\Delta y}{\Delta x} = \frac{f(a+h) - f(a)}{h} \text{ is sometimes called the difference quotient.}$$

It is the average rate of change of y with respect to x over the interval [a, a + h]. The derivative f'(a) can be considered as the instantaneous rate of change of y with respect to x, at x = a.

# Example, and derivative as a function

Example: For $f(x) = x^2$, find the derivative at $x = a$.

Solution: $\dfrac{f(a+h) - f(a)}{h} = \dfrac{(a+h)^2 - a^2}{h} = 2a + h$

$\therefore f'(a) = \lim_{h \to 0} \left(2a + h\right) = 2a + \lim_{h \to 0} h = 2a$

Instead of evaluating the derivative at x = a, we could evaluate it at a generic point x in the domain of f(x). By doing this, the derivative becomes a function (denoted by f'(x)), whose value depends on x. We can then say that f'(a) (as in the above example) is the value of f'(x) at x = a.

So $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

- The process of finding f'(x) is called **differentiation**. The domain of f'(x) is a subset of the domain of f(x).
- If the limit exists at x = a, we say f is **differentiable** (has a derivative) **at x = a**.
- If f'(x) exists at all points in the domain of f(x), we call f(x) a **differentiable function**.
- A function is **differentiable on an open interval**, if it has a derivative at each point in the interval.

# Derivative as a function (continued)

A function is **differentiable on a closed interval** [a, b] if it is differentiable on the open interval (a, b) and at the endpoints, the following one sided limits exist (similar to continuity at an endpoint).

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h}$$

called the right hand derivative at $x=a$.

$$\lim_{h\to 0^-}\frac{f(b+h)-f(b)}{h}$$

called the left hand derivative at $x=b$.

Sometimes we will use the notation Rf'(c) and Lf'(c) to denote the right and left hand derivative respectively at x = c.

Setting z = x + h in the derivative definition, gives an alternative expression for f'(x). When h → 0 (h approaches 0), z – x → 0 or z → x. So we have:

$$f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

When $y=f(x)$, the derivative $f'(x)$ is also denoted as

$$\frac{dy}{dx}\text{ or } y' \text{ or } \frac{df}{dx} \text{ or } \frac{df(x)}{dx} \text{ or } Df(x)$$

$f'(a)$, the value of the derivative at $x=a$, is also denoted as

$$\left.\frac{dy}{dx}\right|_{x=a} \text{ or } \left.\frac{df(x)}{dx}\right|_{x=a}$$

# Derivative examples

Example 1: Let $y = f(x) = mx + c$ where $m$ and $c$ are constants

Derivative $f'(x) = \lim_{h \to 0} \dfrac{\left(m(x+h)+c\right) - \left(mx+c\right)}{h} = \lim_{h \to 0} \dfrac{mh}{h} = m$

Example: Let $y = f(x) = |x|$

For $x > 0$, $f(x) = x$, and using Example 1, $f'(x) = 1$

For $x < 0$, $f(x) = -x$, and using Example 1, $f'(x) = -1$

For $x = 0$, the right hand derivative (limit) is $\lim_{h \to 0^+} \dfrac{|0+h| - |0|}{h} = \lim_{h \to 0^+} \dfrac{h}{h} = 1$

and the left hand derivative (limit) is $\lim_{h \to 0^-} \dfrac{|0+h| - |0|}{h} = \lim_{h \to 0^-} \dfrac{-h}{h} = -1$

Since right hand and left hand derivatives are different, the function is not differentiable at $x = 0$.

Example: $y = f(x) = x^n$, where $n$ is a positive integer

$\dfrac{dy}{dx} = \lim_{z \to x} \dfrac{z^n - x^n}{z - x} = nx^{n-1}$ (using the result in the limits lesson).

# Derivative examples (continued)

Example: $y = f(x) = \sqrt{x}$, where $x \geq 0$

$$f'(x) = \lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0}\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Note $f'(x)$ is defined for $x > 0$, and is undefined at $x = 0$.

# Scenarios when derivative doesn't exist

- Scenario 1: Slope of the curve $y = f(x)$ suddenly changes at a point (as in $y = |x|$ at $x = 0$). Graph will have a "corner" and the right hand and left derivatives are not equal at the point.
- Scenario 2: Slope of the curve approaches infinity (vertical tangent), as in $y = \sqrt{x}$ at $x = 0$.
- Scenario 3: Function is discontinuous. At the point of discontinuity, the right and left hand derivatives will differ. e.g. for the greatest integer function, at integral values of x, the right hand derivative is 0 and the left hand derivative doesn't exist (approaches infinity).

# Relationship between continuity and differentiability

- If a function is discontinuous, it is not differentiable.
- If a function is continuous, it may or may not be differentiable.
  - $y = |x|$ is continuous but not differentiable at $x = 0$.
  - $y = x^2$ is continuous as well as differentiable.
- If a function f(x) is differentiable at x = a, then it is continuous x = a (as proved below).

Proof: We need to show that $\lim_{h\to 0} f(a+h) = f(a)$

$$f(a+h) = f(a) + f(a+h) - f(a) = f(a) + \frac{f(a+h)-f(a)}{h}\cdot h \quad \text{where } h \neq 0$$

$$\therefore \lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a) + \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\cdot h\right)$$

$$= f(a) + \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\cdot \lim_{h\to 0} h = f(a) + f'(a).0 = f(a)$$

# Sum rule for derivatives

Let us now consider the rules for finding the derivative, when functions are combined in some way like addition or multiplication.

Sum rule: If f(x) and g(x) are differentiable functions, then the sum f(x) + g(x) is differentiable at every point where f and g are both differentiable. At these points:

$$\frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

Proof: $\displaystyle \frac{d(f+g)}{dx} = \lim_{h\to 0}\frac{\left[f(x+h)+g(x+h)\right]-\left[f(x)+g(x)\right]}{h}$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right)$$

$$= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \frac{df}{dx} + \frac{dg}{dx}$$

# Constant multiple and difference rule for derivatives

If $f(x)$ is a differentiable function of $x$ and $c$ is a constant, then

$$\frac{d\left(cf(x)\right)}{dx} = c\frac{df}{dx}$$

Proof: $\dfrac{d\left(cf(x)\right)}{dx} = \lim\limits_{h\to 0}\dfrac{cf(x+h)-cf(x)}{h} = \lim\limits_{h\to 0}\left(c\cdot\dfrac{f(x+h)-f(x)}{h}\right)$

$$= \lim_{h\to 0} c\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{df}{dx}$$

If $f(x)$ and $g(x)$ are differentiable functions, then the derivative of their difference is

$$\frac{d\left(f-g\right)}{dx} = \frac{d\left(f+(-1)g\right)}{dx} = \frac{df}{dx} + \frac{d\left(-1\cdot g\right)}{dx}$$

$$= \frac{df}{dx} + (-1)\frac{dg}{dx} = \frac{df}{dx} - \frac{dg}{dx}$$

# Product rule for derivatives

At points where both $f(x)$ and $g(x)$ are differentiable

$$\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$$

Proof: $\displaystyle \frac{d(fg)}{dx} = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$

Subtract and add $f(x+h)g(x)$ to the numerator.

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x)\frac{f(x+h) - f(x)}{h} = f(x)\frac{dg(x)}{dx} + g(x)\frac{df(x)}{dx}$$

Note $\displaystyle \lim_{h\to 0} f(x+h) = f(x)$ since $f(x)$ is differentiable, and hence continuous.

# Quotient rule for derivatives

At points where both $f(x)$ and $g(x)$ are differentiable, and $g(x) \neq 0$

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\dfrac{df}{dx} - f\dfrac{dg}{dx}}{g^2}$$

Proof: $$\frac{d}{dx}\left(\frac{f}{g}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right)$$

$$= \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

Subtract and add $f(x)g(x)$ to the numerator

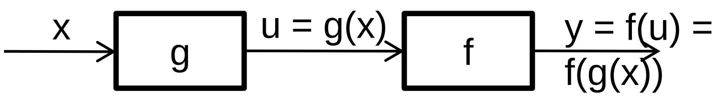
$$= \lim_{h\to 0}\frac{\left[f(x+h)g(x) - f(x)g(x)\right] - \left[f(x)g(x+h) - f(x)g(x)\right]}{hg(x)g(x+h)}$$

$$= \frac{\lim\limits_{h\to 0} g(x)\dfrac{f(x+h) - f(x)}{h} - \lim\limits_{h\to 0} f(x)\dfrac{g(x+h) - g(x)}{h}}{\lim\limits_{h\to 0} g(x)g(x+h)} = \frac{g(x)\dfrac{df(x)}{dx} - f(x)\dfrac{dg(x)}{dx}}{[g(x)]^2}$$

# Chain rule for composite functions

- We now see how to compute the derivative of a composite function. A composite function y = f(g(x)) can be written as y = f(u) where u = g(x).
- If f(u) is differentiable at u = g(x), and g(x) is differentiable at x (corresponding to the value of u), then the composite function y = f(g(x)) is differentiable at x, with the derivative given by:

$$\frac{dy}{dx} = f'\left(g(x)\right)g'(x) = \left.\frac{dy}{du}\right|_{u=g(x)} \times \frac{du}{dx}$$




- To understand this, let x change by Δx. We then have
  - Δu = g(x + Δx) – g(x) and
  - Δy = f(u + Δu) – f(u) where u = g(x)
  - As Δx→0, Δu→0 since u = g(x) is differentiable (and hence continuous) at x.

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right)\frac{du}{dx} = \left.\frac{dy}{du}\right|_{u=g(x)} \frac{du}{dx} = f'(g(x))g'(x)$$

The only flaw in this "proof" is that $\Delta u$ can be 0, even when $\Delta x$ is not (and hence we can be dividing by 0). The actual proof is on the next page, and is optional.

# Chain rule (continued) and examples

Proof (Part 1) [Both parts of proof are optional]

Consider a function $y = f(x)$. Then

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

where $\Delta y = f(x+\Delta x) - f(x)$

Let ε be the difference between the difference quotient and the derivative at (the chosen) $x$,

i.e. $\varepsilon = \dfrac{\Delta y}{\Delta x} - f'(x) \to \Delta y = (f'(x) + \varepsilon)\Delta x$

$$\lim_{\Delta x \to 0} \varepsilon = \lim_{\Delta x \to 0} \left( \frac{\Delta y}{\Delta x} - f'(x) \right) = f'(x) - f'(x) = 0$$

We use the above to prove the chain rule.

Proof (Part 2)

Writing the composite function $y = f\left(g(x)\right)$ as

$y = f(u)$ where $u = g(x)$

$\Delta u = (g'(x) + \varepsilon_1)\Delta x$ where $\varepsilon_1 \to 0$ when $\Delta x \to 0$

$\Delta y = (f'(u) + \varepsilon_2)\Delta u$ where $\varepsilon_2 \to 0$ when $\Delta u \to 0$

$\therefore \Delta y = (f'(u) + \varepsilon_2)(g'(x) + \varepsilon_1)\Delta x$

$$\frac{\Delta y}{\Delta x} = f'(u)g'(x) + f'(u)\varepsilon_1 + g'(x)\varepsilon_2 + \varepsilon_1\varepsilon_2$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u)g'(x) = f'(g(x))g'(x)$$

Example: Let $y = f(x) = (x^2+1)^5$. Find $f'(x)$.

We can write the above as $y = h(u) = u^5$ where $u = g(x) = x^2 + 1$

$$\therefore \frac{dy}{dx} = \left.\frac{dy}{du}\right|_{u=g(x)} \frac{du}{dx} = \left.5u^4\right|_{x^2+1} \cdot 2x = 10x(x^2+1)^4$$

# Examples (continued)

Example: $f(x)=x^n$ where $n$ is a negative integer. Find $f'(x)$.

Let $m=-n$, hence $m$ is a positive integer. Then $f(x)=x^n=\dfrac{1}{x^m}$

Applying the quotient rule for derivatives $f'(x)=\dfrac{x^m\cdot 0-1\cdot mx^{m-1}}{x^{2m}}=-mx^{-m-1}=nx^{n-1}$

Example: Let $y=f(x)=\dfrac{1}{(\sqrt{x^3+5x}+3x^2)^7}$. Find $f'(x)$.

We can write $y=h(u)=\dfrac{1}{u^7}$ where $u=g(x)=\sqrt{x^3+5x}+3x^2$

$\therefore\ f'(x)=-7u^{-8}\Big|_{\sqrt{x^3+5x}+3x^2}\cdot\dfrac{du}{dx}$ (applying chain rule)

$u$ is a sum of the two functions $\sqrt{x^3+5x}$ and $3x^2$, and $\sqrt{x^3+5x}$ is $\sqrt{x}$ composed with $x^3+5x$.

Hence $\dfrac{du}{dx}=\dfrac{1}{2\sqrt{x^3+5x}}\cdot(3x^2+5)+6x$ (chain rule applied to $\sqrt{x^3+5x}$)

$$\therefore\ f'(x)=\frac{-7}{\left(\sqrt{x^3+5x}+3x^2\right)^8}\left(\frac{3x^2+5}{2\sqrt{x^3+5x}}+6x\right)$$

# Examples (continued)

Usually we do the differentiation of a composite function in a single step, starting with the "outermost function" and proceeding inwards as shown below.

$$\text{Example: Let } y = f(x) = \left(\sqrt{x^4 + \frac{2}{x}} + 7x^2 + 1\right)^5$$

$$f'(x) = 5\left(\sqrt{x^4 + \frac{2}{x}} + 7x^2 + 1\right)^4 \times \left[\frac{1}{2\sqrt{x^4 + \frac{2}{x}}} \times \left(4x^3 - \frac{2}{x^2}\right) + 14x\right] = 5\left(\sqrt{x^4 + \frac{2}{x}} + 7x^2 + 1\right)^4 \left(\frac{2x^5 - 1}{x^{3/2}\sqrt{x^5 + 2}} + 14x\right)$$

Note we can write $y = u^5$ (the outermost function), where $u = \sqrt{v} + 7x^2 + 1$ and $v = x^4 + 2/x$.

The calculus of non-algebraic functions is available to those who subscribe to our course. However, by knowing some standard formulae (given below), we should have no problem differentiating functions composed from them.

| Function f(x) | Derivative f'(x) |
|---|---|
| sin x | cos x |
| cos x | −sin x |
| tan x | $\sec^2 x$ |
| cot x | $-\csc^2 x$ |
| sec x | sec x tan x |

| Function f(x) | Derivative f'(x) |
|---|---|
| csc x | −csc xcot x |
| $e^x$ | $e^x$ |
| ln\|x\| | 1/x |
| $a^x$ (where a > 0) | $a^x \ln a$ |
| $\log_a x$ (a > 0, a ≠ 1) | 1/(xlna) |

# Other functions and their derivatives

| f(x) | $\sin^{-1} x$ | $\cos^{-1} x$ | $\tan^{-1} x$ | $\cot^{-1} x$ | $\sec^{-1} x$ | $\csc^{-1} x$ |
|---|---|---|---|---|---|---|
| f′(x) | $\frac{1}{\sqrt{1-x^2}}$ | $\frac{-1}{\sqrt{1-x^2}}$ | $\frac{1}{1+x^2}$ | $\frac{-1}{1+x^2}$ | $\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$ | $\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$ |

Those who have subscribed to our course should know all the above functions very well (we will also prove many of the derivative formulae in a later lesson). For others, a few points are given below.

- Trigonometric functions like sin x are usually defined for all values of angle x. x is measured in radians, where π radians = 180º. For example, x = π/4 implies x = 180º/4 = 45º.
- Functions like $\sin^{-1}x$ are called inverse trigonometric functions. By definition, if y = $\sin^{-1}x$, then x = siny (similarly for the other functions as well). So y is an angle in radians, e.g. $\sin^{-1}1 = \pi/2$. Their domains depend on the function, for example $\sin^{-1}$ and $\cos^{-1}$ are defined only for [−1, 1].
- $e^x$ and $a^x$ are called exponential functions, where e is special irrational number, which is roughly equal to 2.72. The functions are defined for all values of x.
- lnx is the logarithmic function for which the base is e (so ln x = $\log_e x$). It is defined for x > 0.

Example: Find the derivative of f(x) = tanx, using the derivative of sinx and cosx (from the table).

$$f(x) = \tan x = \frac{\sin x}{\cos x}\text{. Using quotient rule,}$$

$$f'(x) = \frac{\cos x \cdot \cos x - (\sin x \cdot -\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

# Examples (continued)

Example: Using the derivative of $\ln|x|$, find $f'(x)$ for $f(x) = \ln(x + \sqrt{x^2+5})$

$$f'(x) = \frac{1}{x+\sqrt{x^2+5}} \times \left(1 + \frac{1}{2\sqrt{x^2+5}} \times 2x\right) = \frac{1}{\sqrt{x^2+5}}$$

The outermost function is ln t, where t = x + $(x^2 + 5)^{1/2}$. Note t > 0 for all x.

Example: Find $f'(1)$ when $f(x) = \frac{1}{a}\tan^{-1}\frac{x}{a}$ where $a$ is a constant.

Using the derivative formula for $\tan^{-1}x$, we have

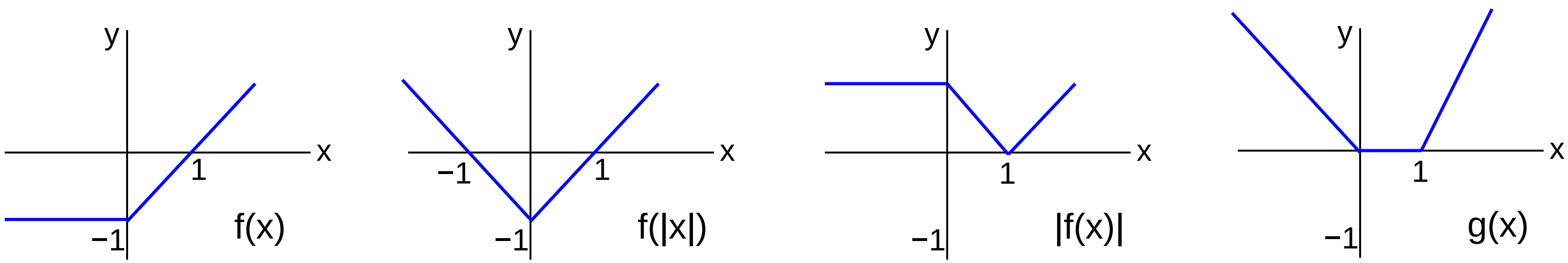
$$f'(x) = \frac{1}{a} \times \frac{1}{1+\left(\frac{x}{a}\right)^2} \times \frac{1}{a} = \frac{1}{a^2+x^2} \rightarrow f'(1) = \frac{1}{a^2+1}$$

# Examples (continued)

[IIT 1986] Let f(x) be defined in the interval [−2, 2] such that
f(x) = −1 in [−2, 0] and
x – 1 in (0, 2].

Let g(x) = f(|x|) + |f(x)|. Determine the differentiability of g(x) in (−2, 2).

Solution: When we have a function like g(x), defined using absolute values of other simple functions like f(x), it is usually easy to sketch g(x) to make our conclusion.




Students of our course should have no problem in understanding the above graphs. However an explanation is given below.

When x ≥ 0, f(|x|) is the same as f(x). For x < 0, f(|x|) = f(−x); so to build the graph for x < 0, we reflect the graph for x > 0 in the y-axis.

|f(x)| reflects the negative part of f(x) in the x-axis, while retaining the positive part of f(x).

g(x) = −x for [–2, 0) and 0 for [0, 1) and 2x – 2 for [1, 2].

From the graph of g(x), it is obvious that it is not differentiable at x = 0, 1. Everywhere else in (−2, 2), it has a derivative.

# Higher order derivatives and an example

Let $y = f(x)$ and $f'(x)$ be its derivative.

Since $f'(x)$ is a function, it can be differentiated to give the second derivative of $f$.

Various notations for the 2nd derivative are $f''(x) = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{dy'}{dx} = D^2 f(x) = y''$

Similarly $f''(x)$ is a function that can be differentiated again and this process can be repeated.

The $n$th derivative is written as $\dfrac{d^ny}{dx^n} = \dfrac{d}{dx} y^{(n-1)} = y^{(n)} = D^n f(x)$

Example: Let $y = f(x) = x^4$. We then have

$$\frac{dy}{dx} = 4x^3 \qquad \frac{d^2y}{dx^2} = 12x^2$$

$$\frac{d^3y}{dx^3} = 24x \qquad \frac{d^4y}{dx^4} = 24$$

All the other higher derivatives are 0.

# Implicit differentiation

- Consider the equation $y^4 - 4xy^3 + x^2 = 5$.
- We cannot put it in the form $y = f(x)$ to find $dy/dx$. However, we can consider it as implicitly defining one or more functions of x and proceed as follows.
  - Differentiate both sides with respect to x.
  - Treat y as a differentiable function of x.
  - Any function of y such as $y^4$ is differentiated using the chain rule.
  - Collect all terms involving dy/dx together, and solve the equation for dy/dx.

Example: Find $dy/dx$ for $y^4 - 4xy^3 + x^2 = 5$.

Differentiating both sides with respect to $x$

$$4y^3 \frac{dy}{dx} - 4\left( y^3 + x \cdot 3y^2 \frac{dy}{dx} \right) + 2x = 0 \to \left(2y^3 - 6xy^2\right)\frac{dy}{dx} = 2y^3 - x \to \frac{dy}{dx} = \frac{2y^3 - x}{2y^2(y-3x)}$$

Example: Find $dy/dx$ for $y = x^{p/q}$ where $p/q$ is a rational number

$y = x^{p/q} \to y^q = x^p$. Differentiating implicitly, we get $qy^{q-1}\dfrac{dy}{dx} = px^{p-1}$

$$\to \frac{dy}{dx} = \frac{px^{p-1}}{qx^{(p/q)(q-1)}} = \frac{px^{p-1}}{qx^p \cdot x^{-p/q}} = \frac{p}{q}x^{p/q-1}$$

Hence the derivative formula for $x^n$ (which is $nx^{n-1}$) is valid for rational powers of $x$ also.

# Implicit differentiation examples (continued)

Example: The equation $x^2 + y^2 = 36$ defines two functions $y_1 = \sqrt{36 - x^2}$ and $y_2 = -\sqrt{36 - x^2}$. The equation is a circle with $y_1$ representing the part above $x$ axis, while $y_2$ representing the part below. Differentiating implicitly, we have

$2x + 2y\dfrac{dy}{dx} = 0 \rightarrow \dfrac{dy}{dx} = \dfrac{-x}{y}$. This expression for derivative applies to both $y_1$ and $y_2$.

The above result is consistent with what we can conclude from coordinate geometry. Since the circle center is at the origin, the slope of a radial line to the point (x, y) on the circumference is y/x. The tangent is perpendicular to the radius, so its slope is –x/y (same as dy/dx above).

Example : We will see in a later lesson that a second degree equation in x and y given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a conic section (including degenerate forms like a pair of straight lines). Find the slope of the tangent drawn at a point (x, y) on the curve.

Solution: Tangent slope at (x, y) is the value of dy/dx at this point. Differentiating implicitly, we have

$$2ax + 2hy + 2hx\frac{dy}{dx} + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f}$$

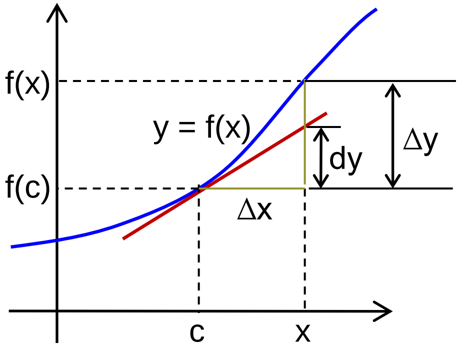
# Differential

- Consider the function $y = f(x)$. The **differential of y** denoted by $dy$, is <u>defined</u> as $dy = f'(x)\Delta x$.
  - The value of $dy$ depends on $x$ <u>and</u> $\Delta x$.
  - Since $y = f(x)$, we may also write $d(f(x)) = f'(x)\Delta x$, where $d(f(x))$ is called the differential of f.
  - When $y = f(x) = x$, we have $f'(x) = 1$, and $dy = \Delta x$. Since $y = x$, we can also write $dy = dx = \Delta x$. The differential $dx$ of the independent variable x, is the same as its increment (and this is sometimes taken as a definition).
  - So for the generic function $y = f(x)$, we may write $dy = f'(x)dx$.
- To interpret the differential $dy$, let us take $x = c$. We then have $dy = f'(c)\Delta x$ (1)
  - Consider the tangent through $(c, f(c))$. Its equation (point slope form) is: $y - f(c) = f'(c)(x - c)$ (2)
  - Using 1) and 2), when $\Delta x = x - c$, we have $dy = y - f(c)$. So $dy$ is the change in y on the tangent line, when x changes from c by $\Delta x$ (see figure).

The tangent line at $x = c$, approximates $f(x)$ in the vicinity of c.

So $f(x) \cong f(c) + f'(c)(x - c)$ when x is close to c.

It is called a *linear approximation* to $f(x)$ at $x = c$ (linear, because only the first power of $x - c$ appears).



# Example

Example: Use differentials to evaluate $\sqrt[4]{82}$.

Solution: Let $f(x)=\sqrt[4]{x}$. We want $f(82)$, and we know $f(81)=3$. Using differentials, implies using the linear approximation at $x=81$ to evaluate $f(82)$.

We have $f'(x)=\dfrac{1}{4x^{3/4}} \rightarrow f'(81)=\dfrac{1}{4\times 81^{3/4}}=\dfrac{1}{108}$

$f(82)\cong f(81)+f'(81)\left(82-81\right)=3+1/108=3.009.$

Suppose y = f(x), where x = g(t).

We then have y = f(g(t)), and the differential dy = f'(g(t))g'(t)dt (1)

Also the differential dx = g'(t)dt. Substituting in 1), we get dy = f'(x)dx (2)

In Eq 2), it is incorrect to replace dx by Δx. However, if x is the independent variable, then we can replace dx by Δx (as seen earlier).

The above means that Eq 2) between the differentials of y and x is always valid, whether or not x is the independent variable. On the other hand, the relation dy = f'(x)Δx is valid, <u>only</u> when x is the independent variable.